On the stability of a RLC parametric oscillator

Luis Moreno-Ahedo*, Marisol Prado*, Roberto Carmona** and Armando Aguayo*

Unidad de Estudios de Posgrado e Investigación*
Tecnológico de Estudios Superiores de Coacalco
lmoreno@ieee.org, marisolprado@ieee.org, webmaster@tesco.edu.mx
Universidad Politécnica del Valle de México
rcarmona@ctrl.cinvestav.mx

Abstract. We analyze the stability of a RLC parametric oscillator when the frequency and amplitude (γ, Ω) of the excitation source varies, the oscillator is modeled as a parametrically excited system then Floquet theory approach is used to study the stability by stating the monodromy matrix which is approximated symbolically, the analysis show in the (γ, Ω)-plane the dominoes of stability and instability, a propose algorithm computes the transition curves that is the boundaries between the dominoes of stability, the analysis is supported by numerical simulations.

Keywords: Parametric oscillator, Arnold's tongues, monodromy matrix, parametrically excited systems, parametric resonance

1 Introduction

When the model of system the applied excitation source appears as a time-varying coefficient the system is called parametrically excited (PE) system [1], examples of such systems are: the variable length pendulum and the pendulum with a vertically oscillating pivot, torsional oscillators, cantilevers and translational oscillators to name a few ones.

Quite time ago Melde [2] and Faraday [3] recognized experimentally the parametric resonance phenomenon occurred in PE systems, through the work developed by [4], [5], [6], [7] and [8] now it is well-know that PE systems experience parametric resonance when driven at frequencies close to twice ω_0/n, where $n \leq 1$ and ω is the natural frequency, in many fields of engineering this phenomenon can occur, for example in bridges [9] and cause degradation and failure of mechanical structures, however some current applications exploiting parametric resonance for example mass sensing [10] and signal filtering [11].

In the study of PE systems there are well established results through three basic approaches: the perturbation theory [12], the harmony balance [13] and the Floquet theory [14]. However the study of PE system with varying frequency has received little attention namely, in [15] the resonance zones are determinate by approximation procedures and in [16] the harmony balance is used to analyse a PE system with combined frequencies.

The problem of studying a PE system with varying frequency can be seen as the frequency analysis of linear time periodic systems where the main results are [17] which

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used an one-to-one map induced by geometrically periodic signals and [18] where a general harmony balance method is used to stablish the frequency response. We shall apply the algorithm developed in [19] to analysis the stability of a parametric oscillator when the frequency of the excitation source varies, the oscillator is implemented as the parametric forced RLC circuit shown in figure 1.

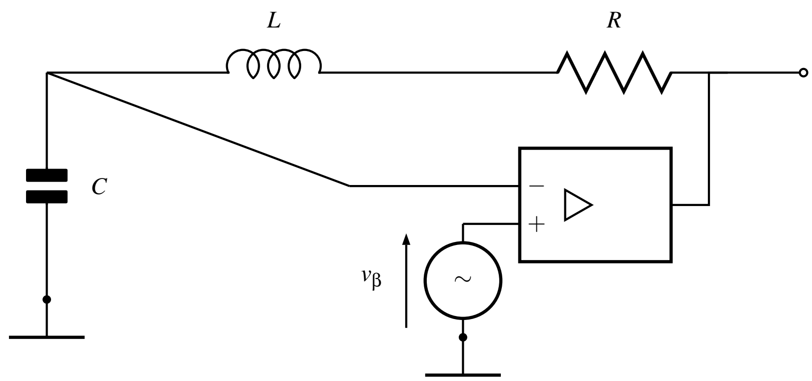


Fig. 1. Circuit of a RLC parametric oscillator, the box represent an analog multiplier

2 RLC parametric oscillator model

In this section the RLC parametric oscillator [20] is modeled. Applying the Kirschoff's voltage law to the electronic circuit shown in the figure 1 yields

$$v_C + v_R + v_L = k v_C v_\beta$$

where the voltages are given by $v_R = R\frac{dq}{dt}$, $v_C = \frac{q}{C}$, $v_L = L\frac{d^2q}{dt^2}$ and $v_\beta = \beta \cos \omega t$, substituting and ordering terms

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}(1 - k\beta \cos \omega t)q = 0$$

where q is charge, C capacitance, L inductance, R resistance, k the gain of the analog multiplier and the pair (γ, ω) are the amplitude and frequency of the excitation source v_β, without loss of generality can be written

$$\ddot{q} + \lambda \dot{q} + \omega_0^2(1 + k\beta \cos \omega t)q = 0$$

where $\omega_0^2 = \frac{1}{LC}$ and $\lambda = \frac{R}{L}$, the above equation using the dimensionless variable $\tau = \omega_0 t$ can be reduce to:

$$\frac{d^2q}{d\tau^2} + \mu \frac{dq}{d\tau} + (1 + \gamma \cos \Omega \tau)q = 0 \tag{1}$$

where $\Omega = \frac{\omega}{\omega_0}$, $\gamma = k\beta$ and $\mu = \frac{R/L}{\omega_0}$.

The change of variable, [21]:

$$q = xe^{\frac{1}{2}\mu}$$

transforms the equation (1) into:

$$\ddot{x} + \left(1 - \frac{1}{4}\mu^2 + \gamma\cos\Omega\tau\right)x = 0 \tag{2}$$

For a given dissipative term μ, we shall plot the stability chart in the (γ,Ω)-plane of Eq. (2).

The above equation corresponds to so-called Mathieu equation, [22]:

$$\ddot{x} + [\alpha + \beta\cos(t)]x = 0$$

which is the a special case of the Hill's equation, [22]:

$$\ddot{x} + [\alpha + \beta p(t)]x = 0$$

where $p(t+T) = p(t)$ and $\int_0^T p(t) = 0$

For both equations the stability is usually given in terms of the parameters as a stability chart in the (α,β)-plane as the figure 2 shows, the shaded regions are zones where the parametric resonance occurs such zones are known as resonance zones or Arnold's tongues [23], while the white regions are zones where the solution is bounded, the boundaries are called transition curves where the solution is periodic. To analyse this kind of equations it is used the Floquet theory given in following section.

3 Preliminaries

Consider the linear time periodic (LTP) system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A(t)x}(t) \tag{3}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. The Floquet theory states, [14]:

Theorem 1. (Floquet's theorem) *Any* $\boldsymbol{\Phi}$ *fundamental matrix of* (3) *can be written as:*

$$\boldsymbol{\Phi}(t) = \boldsymbol{P}(t)e^{\boldsymbol{R}t}$$

where $\boldsymbol{R},\boldsymbol{P} \in \mathbb{R}^{n\times n}$ *is non-singular,* $\boldsymbol{P}(t) = \boldsymbol{P}(T+t)$ *and* T *is the minimal period* □

Floquet theory also shows that $\boldsymbol{\Phi}(t+T) = \boldsymbol{\Phi}(t)e^{\boldsymbol{R}T}$, if $t = 0$ then:

$$\boldsymbol{C} \triangleq e^{\boldsymbol{R}T} = \boldsymbol{\Phi}^{-1}(0)\boldsymbol{\Phi}(T)$$

The matrix $\boldsymbol{C}$ is the so-called monodromy matrix [13], its eigenvalues λ_i are called characteristic multipliers. The importance of the matrix is shown in the following corollary.

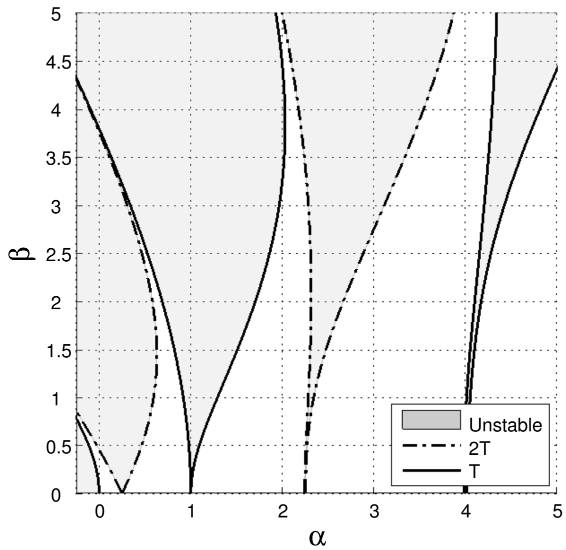


Fig. 2. Stability chart of the Mathieu eq. $\ddot{x}+(\alpha+\beta\cos t)x=0$

Corollary 1. *The solution* $\boldsymbol{x}(t)$ *of the system* (3) *satisfies*

$$\boldsymbol{x}(t+T)=\lambda\boldsymbol{x}(t)$$

if and only if λ *is a characteristic multipliers of* (3). □

From the above corollary we deduce the criterion shown in table 1.

The above result is elegant but computing the characteristic multiplier in practical cases is almost impossible however the Floquet theorem applied to the case of Hill's equation provides an attractive result as follows.

Solutions	Multipliers
Stability trivial solution	Inside or on the unit circle (simple multipliers)
Asymptotic	Inside unit circle
Instability of of trivial solution	At least one outside the unit circle or on the unit circle with multiple elementary divisor
Periodic solution	At least one multiplier equal to 1 or -1

Table 1. Multipiers

The Hill's equation written in state variable yields:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\alpha - \beta p(t) & 0 \end{bmatrix} \boldsymbol{x} = \boldsymbol{A}(t)\boldsymbol{x} \tag{4}$$

Let be $\boldsymbol{\Phi}$ a fundamental matrix of the system (4) and initial conditions such that $\boldsymbol{\Phi}(0) = \boldsymbol{I}$. Then the monodromy matrix of (4) is $\boldsymbol{C} = \boldsymbol{\Phi}(T)$. The characteristic multipliers are given by $p(\lambda) = \lambda^2 - \text{tr}[\boldsymbol{C}]\lambda + \det[\boldsymbol{C}]$ applying the Liouville's formula[1] yields $\det[\boldsymbol{C}] = \det[\boldsymbol{\Phi}(T)] = 1$ then $\lambda_i = \frac{1}{2}\left[\phi \pm \sqrt{\phi^2 - 4}\right]$ where $\phi = \text{tr}[\boldsymbol{C}] = \text{tr}[\boldsymbol{\Phi}(T)]$ is the trace of the monodromy matrix applying the results of table 1 we deduce the following criterion.

Criterion 1 *Let be $tr[\boldsymbol{C}] = tr[\boldsymbol{\Phi}(T)]$ the trace of the monodromy matrix and $x(t)$ the solution of* (4) *then, [25]:*

(i) If $|\text{tr}[\boldsymbol{\Phi}(T)]| < 2$ then $x(t)$ is bounded.
(ii) If $|\text{tr}[\boldsymbol{\Phi}(T)]| > 2$ then $x(t)$ is unbounded.
(iii) If $|\text{tr}[\boldsymbol{\Phi}(T)]| = 2$ then $x(t)$ is periodic □

This criterion enable us to evaluate only the trace of the monodromy matrix $\text{tr}[\boldsymbol{\Phi}(T)]$ instead of calculating the characteristic multipliers, which is practical because the trace $\text{tr}[\boldsymbol{\Phi}(T)]$ can be approximated using the algorithm developed in [19] which approximates symbolically the monodromy matrix using the Taylor's method for ordinary differential equations [26], the algorithm is implemented in a computer algebra program, namely Mathematica©.

4 Computing transition curves

A pseudo-code of the algorithm to approximate the monodromy matrix is given in Algorithm 4.1.

The inputs of the algorithm are: the periodic matrix $A(t)$, the minimal period T , M is order Taylor's method, n is the number of divisions of the time interval and m is the matrix A dimension.

The first loop (i) in the algorithm computes the general step of the method, the second (ii) loop makes a copy of the general step to compute the vectors column solutions $\boldsymbol{x}_i$ of the monodromy matrix and approximates such solutions as a sequence of points, the line (iii) takes the last element of the sequences to form the monodromy matrix. In the algorithm $\boldsymbol{e}_i$ are the standard basis of the vector space $\mathbb{R}^m$.

[1] [24] Let bet $\boldsymbol{\Phi}(t)$ a fundamental matrix of $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$ then $\det|\boldsymbol{\Phi}(t)| = \det|\boldsymbol{\Phi}(0)|\, e^{\int_{t_o}^{t} \sum_{j=1}^{n} a_{jj}(s)ds}$

Algorithm 4.1: MONODROMYMATRIX($\boldsymbol{A}(t), T, M, n, m$)

$\boldsymbol{F} := \boldsymbol{A}(t)\boldsymbol{x}_j^i[t]; h := \frac{T}{n}; \boldsymbol{S} := \boldsymbol{x}_j^i[t]$
for $l := 1$ **to** M
  do (i)
$\begin{cases} \boldsymbol{F} := \frac{d\boldsymbol{F}}{dt} \\ \boldsymbol{S} := \boldsymbol{S} + \frac{h^l}{l!}\boldsymbol{F} \end{cases}$ t:=0;
for $i := 1$ **to** m
  do (ii)
$\begin{cases} \boldsymbol{S}_i := \boldsymbol{S} \\ \boldsymbol{x}_j^i[0] = \boldsymbol{e}_i \\ \textbf{for } j := 1 \textbf{ to } n \\ \quad \textbf{do} \\ \left\{ \boldsymbol{x}_j^i[t+h] := \boldsymbol{S}_i[t] \Big|_t \right. \\ t := t + h \end{cases}$
$\boldsymbol{C} := \left[\boldsymbol{x}_n^1 \; \boldsymbol{x}_n^2 \; \cdots \; \boldsymbol{x}_n^m \right]$ (iii)

Writing the equation of the electronic parametric oscillator (2) in state variable yields:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\left(1 - \frac{1}{4}\mu^2 + \gamma \cos \Omega\tau\right) & 0 \end{bmatrix} \boldsymbol{x} = \boldsymbol{A}(t)\boldsymbol{x}$$

Suppose that the dissipative term is $\mu = 1$ then we shall approximate the monodromy matrix $\boldsymbol{C}$ with the algorithm 4.1 for each frequency Ω between $(0.25, 1.5)$ with a step $h = 0.01$ considering γ as a parameter, for example considering $\Omega = 1$ the algorithm approximates the trace of the monodromy matrix $\phi = \mathrm{tr}[\boldsymbol{C}]$ as a polynomial

$$\begin{aligned} &-1.95 - (1.61 \times 10^{-15})\gamma + 1.08\gamma^2 - (3.5 \times 10^{-15})\gamma^3 + 6.44\gamma^4 \\ &+4.77\gamma^6 + (1.12 \times 10^{-14})\gamma^7 + 2.14\gamma^8 + (1.11 \times 10^{-15})\gamma^9 + 1.99\gamma^{10} \\ &\vdots \\ &-(1.44 \times 10^{-19})\gamma^{21} \cdots - (1.86 \times 10^{-84})\gamma^{67} + (6.36 \times 10^{-73})\gamma^{68} \\ &-(3.73 \times 10^{-80})\gamma^{70} - (4. \times 10^{-96})\gamma^{71} + 0.\gamma^{72} \end{aligned}$$

The γ roots of the polynomial $|\phi| = 2$ are the values for which the solution of the equation (2) is periodic according with the criterion 1.

In this way computing for each $\Omega \in (0.25, 1.5)$ the γ values for which the solution of (2) is periodic we shall get the transition curves in the (γ, Ω)-plane, this curves are the boundaries between stable an unstable dominoes, the result of this procedure is show in the figure 3 where the mark $\boldsymbol{U}$ is for unstable dominoes and the mark $\boldsymbol{S}$ is for stable dominoes.

In the stability chart of figure 3 the unstable dominoes are regions where the parametric resonance phenomenon occurs that is where the solution of the equation (2)

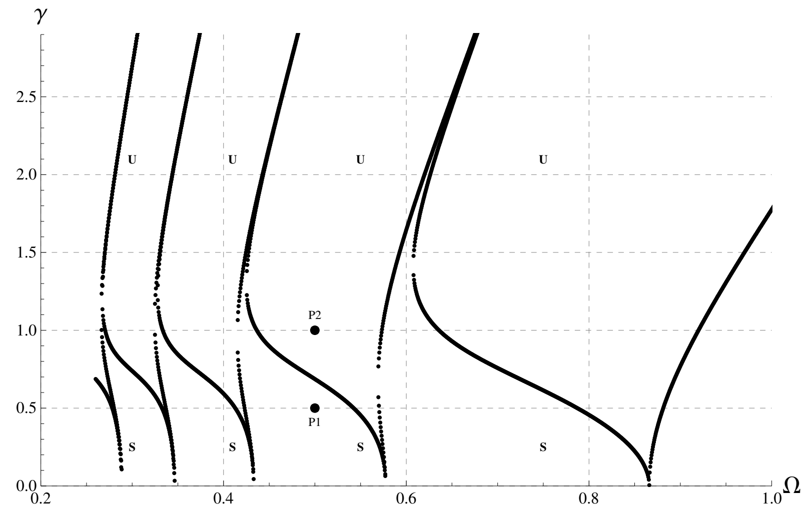
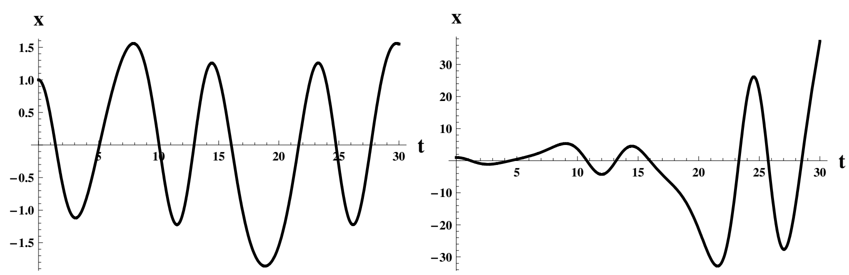


Fig. 3. Stabillity chart in the plane (γ,Ω)-plane of the parametric oscillator (2)



(a) $P_1 = (\gamma,\Omega) = (0.5, 0.5)$ (b) $P_2 = (\gamma,\Omega) = (1, 0.5)$

Fig. 4. Solution of the equation $\ddot{x} + \left(1 - \frac{1}{4}\mu^2 + \gamma\cos\Omega\tau\right)x = 0$ for $\mu = 1$

grows exponentially, for example consider the solution of the eq. (2) for the cases shown in chart 3, namely $P_1 = (\gamma,\Omega) = (0.5, 0.5)$ and $P_2 = (\gamma,\Omega) = (1, 0.5)$, the solutions are plotted in the figure 4 as can been seen the solution for P_1 is bounded while for P_2 grows exponentially.

As final remark the actual Mathematica code which computes the stability chart of the equation 2 is given in the appendix.

5 Conclusions

The stability analysis for a RLC parametric oscillator when the frequency and the amplitude (Ω,γ) of the excitation source varies is presented by approximating symbolically the monodromy matrix in terms of the parameter γ for each discrete step-size of the frequency Ω, the approximation enable to compute the transition curves in the (Ω,γ)-plane these are the boundaries between the stable and unstable dominoes, the actual stability chart is given and supported by numerical simulations of the system's response.

Appendix

The following Mathematica© code have been developed to run with version 6.0.

```
In[9]:= Clear[S1, η, ω, β, M, S, m, F, α, f, γ, A, P, q, g, k, k1, L,
         k2, η, β, α, n, t, M, Periodo]

In[10]:= LISTA = {};
        For[m = 0.5, m ≤ 10, {Periodo = 2 * Pi / m; p = 1 - 1/4 + γ * Cos[m * t];
          A = N[Expand[( 0  1 ; -(p)  0 )]]; M = 6; n = 32; H = N[Periodo/n];
          F = A.( x1[t] ; x2[t] ); {( x1'[t] ; x2'[t] ) = F, f = F, f = Integrate[f, t],
           S = ( x1[t] ; x2[t] )};
          For[i = 1, i ≤ M, {f = Expand[D[f, t]],
            S = TrigReduce[H^i / Factorial[i] * f + S]}; i++];
          S1 = ReplaceAll[S, {x1[t] → x11[t], x2[t] → x22[t]}];
          {( x1[0] ; x2[0] ) = ( 1 ; 0 ), ( x11[0] ; x22[0] ) = ( 0 ; 1 )}; T = 0;
          For[i = 1, i ≤ n, {( x1[H + T] ; x2[H + T] ) = ExpandAll[ReplaceAll[S, {t → T}]],
            ( x11[H + T] ; x22[H + T] ) = ExpandAll[ReplaceAll[S1, {t → T}]], T = T + H},
           i++]; ϕ = Expand[x1[T] + x22[T]]; Blimite = 15; Fb = ϕ - 2;
          Ra1 = Solve[Fb == 0, γ, VerifySolutions→ True]; Fb = ϕ + 2;
          Ra2 = Solve[Fb == 0, γ, VerifySolutions→ True];
          Sol = Sort[Join[Ra1, Ra2]]; Ra = γ /. Sol;
          Ra = Select[Ra, (Im[#] == 0 && Re[#] < Blimite && Re[#] > 0) &];
          For[k = 1, k ≤ (Length[Ra]),
           {LISTA = AppendTo[LISTA, {m, Extract[Ra, k]}]}; k++],
          Print[m]}; m = m + 0.01]

In[16]:= P = ListPlot[LISTA, PlotRange -> {{-1, 5}, {0, 3}}]
```

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